Generalizing Engel Sinks To Other Algebraic Structures

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Abstract

In this paper, we will be proposing a generalization of the theory of Engel sinks to other algebraic structures, of which the original theory for groups was developed predominantly by P. Shumyatsky and E.I. Khukhro, following a series of 8 papers together.

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1 Introductory Theory

Def. 1.1. *A "right Engel sink", $\mathscr{E}_{right}(g)$ for an element, g, in a group, G, is a set such that it contains all sufficiently long commutators* $[[...[g,x],x,]...],x]$.[KS20]
Def. 1.2. *Similarly, a "left Engel sink", $\mathscr{E}_{left}(g)$, is the set which contains all sufficiently long commutators,* $[[...[x,g],g,]...],g]$.[KS20]

2 Results of Research

Engel sinks, as defined by Khukhro and Shumyatsky, are traditionally for groups, and have been used to study subjects such as profinite groups and nilpotency, such as in KS17 where they study finite groups having bounded-rank Engel sinks.
For our approach, we will consider S to be any invertible algebraic structure, which is "at least" a quasigroup. That is, it is required to have invertibility and closedness, but associativity, identity, commutativity, and any other properties are optional. However, the identity property will be critical in defining a general

connection between Engel sinks and nilpotency. Similarly to how we would define the commutator for a group, G, we define the commutator for an arbitrary S, of s_1, s_2 both in S to be $s_1 s_2 s_1^{-1} s_2^{-1}$, equivalent to the definition for a group. For an algebraic structure with 2 operators, $\star_1$ and $\star_2$, there are 2 types of Engel sinks we can define, but they will not be defined separately in the following definitions. Instead, we will use # for a generic operator.
Now, we summarize this in the two following definitions:
Def 2.1 *A "right generalized Engel #-sink", $\mathscr{E}_{right}(s)$ for an element, m, in an algebraic structure S is the set, for every element $x \in S$ containing all sufficiently long commutators $[[...[s,x]_\#, x,]_\#...]_\#, x]_\#$. Here, $[x,y]_\# \triangleq x^{1\#}y^{1\#}x^{-1\#}y^{-1\#}$.*
Def. 2.2. *A "left generalized Engel #-sink", $\mathscr{E}_{left}(s)$, similarly is the set, for every element $x \in S$ containing all sufficiently long commutators $[[...[s,x]_\#, x,]_\#...]_\#, x]_\#$.*
Thm 2.1. *When $Var(S) = \mathfrak{C}_{Grp}$, where $\mathfrak{C}_{Grp}$ is the class of all groups, our definitions reduce to those of **Def 1.1** and **Def 1.2** (1).*
Proof. `This is trivial to prove.`
There is an observation we can make, regarding the connection between nilpotency and Engel sinks – that is, we can create a "nilpotency condition" for any algebraic structure which is "at least" a loop, analogously to how Khukhro and Shumyatsky established. Formally:
Def 2.3. *An algebraic structure, S, is nilpotent if the generalized Engel sinks of all of its elements are equal to the identity.*
To further generalize this condition, we may relax it slightly. Instead of having them eventually equal the identity, we let them eventually become periodic within a small subset of S, and with a small period. What exactly defines "small", is, ideally, $\leq \lceil \sqrt{|S|} \rceil$. This includes the original condition, because a function which is constant beyond some point can be thought of as periodic beyond that point, since a constant function satisfies $f(x+m) = f(x) \forall m \in dom(f)$. Or, in this case, a constant subfunction. This allows us to somewhat improve on the scope of our function, since it doesn't require us to explicitly have an identity defined.

3 Works Referenced/Utilized

[KS20] Compact groups in which all elements have countable Engel sinks – E.I. Khukhro, P. Shumyatsky
[KS17] Fin. groups with Engel sinks of bounded rank – E.I. Khukhro, P. Shumyatsky